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Intermediate Symmetries In Electronic Systems: Dimensional Reduction, Order Out Of Disorder, Dualities, And Fractionalization

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We discuss symmetries intermediate between global and local and formalize the notion of dimensional reduction adduced from such symmetries. We apply this generalization to several systems including liquid crystalline phases of Quantum Hall systems, transition metal orbital systems, frustrated spin systems, (p+ip) superconducting arrays, and sliding Luttinger liquids. By considering space-time reflection symmetries, we illustrate that several of these systems are dual to each other. In some systems exhibiting these symmetries, low temperature local orders emerge by an "order out of disorder" effect while in other systems, the dimensional reduction precludes standard orders yet allows for multiparticle orders (including those of a topological nature).

Keywords:

1. Introduction and Main Results

Orders are often very loosely classified into two types:

(i) *Global symmetry breaking orders*. In many systems (e.g. ferromagnets), there is an invariance of the basic interactions with respect to global symmetry operations (e.g. rotations in the case of ferromagnets) simultaneously performed on all of the fields in the system. At sufficiently low temperatures (or high coupling), such symmetries may often be "spontaneously" broken.

(ii) *"Topological orders"*¹. In some cases, even if global symmetry breaking cannot occur, the system still exhibits a robust order of a topological type. This order may

only be detected by non-local correlation functions. The most prominent examples of this order are afforded by gauge theories which display local gauge symmetries.

In this article, we investigate *intermediate* (or sliding) symmetries which, generally, lie midway between the global symmetries and local gauge symmetries extremes and provide conditions under which local orders cannot appear. We will review:

(1) a theorem² dictating that in many systems displaying intermediate symmetries, local orders are impossible. This theorem also gives upper bounds on multi-particle correlators and suggests for fractionalization in certain instances.

(2) how symmetry allowed orders in such highly degenerate systems may be stabilized by entropic fluctuations. (This is often referred to as the “order out of disorder” mechanism.) In classical (*large S*) renditions of quantum orbital systems, orbital order is stabilized by thermally driven entropic fluctuations.^{3 4} These classical tendencies may be fortified by the incorporation of zero point quantum fluctuations.

(3) a duality between two prominent systems exhibiting intermediate symmetries. A route by which this duality may be established sheds light on some dualities as direct consequences of geometrical (\mathbb{Z}_2) reflections in space-time.⁵

The results reported here appeared, in full detail, elsewhere. Our aim is to give a flavor of these which the reader may then peruse in detail. For pedagogical purposes, we review in section (2), old results concerning lattice gauge theories. In sections thereafter, we discuss new results and constructs concerning systems with non-local intermediate (or sliding) symmetries where some analogies and extensions may be drawn vis a vis the physics of gauge theories which display stronger local symmetries.

2. What are gauge theories and local gauge symmetries?

Throughout this work, we will consider theories defined on a lattice Λ . We start with a review of a very well known topic. In matter coupled gauge theories,^{6 7 8 9} matter fields ($\{\sigma_i\}$) reside at sites i while gauge fields U_{ij} lie on links between sites i and j . The \mathbb{Z}_2 matter coupled to \mathbb{Z}_2 gauge field theory is the simplest such theory. On a hypercubic lattice, its action is

$$S = -\beta \sum_{\langle ij \rangle} \sigma_i U_{ij} \sigma_j - K \sum_{\square} UUUU. \quad (1)$$

The first sum is over all nearest neighbor links $\langle ij \rangle$ while the second is the product of the four gauge fields $U_{ij}U_{jk}U_{kl}U_{ki}$ over each minimal plaquette (square) of the lattice. Both matter (σ_i) and gauge (U_{ij}) fields are Ising variables within this theory: $\sigma_i = \pm 1$, $U_{ij} = \pm 1$. The action S is invariant under local \mathbb{Z}_2 gauge transformations

$$\sigma_i \rightarrow \eta_i \sigma_i, \quad U_{ij} \rightarrow \eta_i U_{ij} \eta_j \quad (\eta_i = \pm 1) \quad (2)$$

There is a theorem, known as Elitzur’s theorem,¹⁰ which disallows quantities not invariant under such these local symmetries (e.g. U, σ) to attain finite expectation values, $\langle \sigma_i \rangle = \langle U_{ij} \rangle = 0$. In a pure gauge theory having no matter coupling ($\beta = 0$ in Eq.(1)), the only symmetry invariant quantities are non-local quantities of a topological nature- the products of gauge fields around closed loops- the “Wilson

loops" $W \equiv \langle U_{ij} U_{jk} \dots U_{pi} \rangle$.⁶ In a pure gauge theory, the asymptotic scaling of W with the loop size dictates what phase we are in.⁶ In the presence of matter, we also find symmetry invariant open string correlators (e.g., $\langle \sigma_i U_{ij} U_{jk} \dots U_{pm} \sigma_m \rangle$).^{6 7}
^{8 9} The simplest realization of Eq.(1) (that in two dimensions) exhibits a non-local (percolation) crossover as a function of β and K .⁹

Electromagnetism is a gauge theory with a local $U(1)$ invariance,

$$\sigma_i \rightarrow \eta_i^* \sigma_i, \quad U_{ij} \rightarrow \eta_i^* U_{ij} \eta_j, \quad (\eta_i = e^{i\theta_i}, \quad \theta_i \in \mathbb{R}). \quad (3)$$

Here, we set $U_{ij} = \exp[iA_{ij}]$, with $A_{ij} = \int_i^j \vec{A} \cdot d\vec{r}$, where \vec{A} is the vector potential. With the complex $U(1)$ fields U_{ij} and σ_{1i} , Eq.(1) is changed by the addition of a complex conjugate. Here, the plaquette term reads $[-\frac{K}{2}(U_i U_{jk} U_{kl} U_{li} + c.c.)] = [-K \cos \Phi_\square]$ with $\Phi_\square = A_{ij} + A_{jk} + A_{kl} + A_{li}$ the flux piercing the plaquette. Expanding, in the continuum limit, $(-K \sum_\square \cos \Phi_\square) \rightarrow (\frac{K}{2} \int dV (\nabla \times \vec{A})^2)$, and familiar continuum electromagnetism appears. Similarly, the first term of Eq.(1) (now $-\frac{\beta}{2} \sum_{\langle ij \rangle} (\sigma_i^* U_{ij} \sigma_j + c.c.)$) becomes, in the continuum, the standard minimal coupling term between charged matter and electromagnetic fields. Here, the Wilson loop becomes the Aharonov-Bohm phase.¹¹ Higher order groups ($U(1) \times SU(2)$, $SU(3)$) describe the electroweak and strong interactions.

3. What are intermediate symmetries?

An intermediate d -dimensional symmetry of a theory² characterized by a Hamiltonian H (or action S) is a group of symmetry transformations such that the minimal non-empty set of fields $\phi_{\mathbf{i}}$ changed by the group operations occupies a d -dimensional subset $\mathcal{C} \subset \Lambda$. The index \mathbf{i} denotes the sites of the lattice Λ . For instance, if a spin theory is invariant under flipping each individual spin then the corresponding gauge symmetry will be zero-dimensional or local. Of course flipping a chain of spins is also a symmetry, but the chain is not the minimal non-trivial subset of spins that can be flipped. In general, these transformations can be expressed as:

$$\mathbf{U}_{lk} = \prod_{\mathbf{i} \in \mathcal{C}_l} \mathbf{g}_{\mathbf{i}k}, \quad (4)$$

where \mathcal{C}_l denotes the subregion l , $\mathcal{C}_l \subset \Lambda$, and $\Lambda = \bigcup_l \mathcal{C}_l$. To make contact with known cases, the local gauge symmetries of Eqs.(2, 3) correspond to $d = 0$ as the region where the local gauge symmetries operate is of dimension $d = 0$. Similarly, e.g. in a nearest neighbor ferromagnet on a D - dimensional lattice, $H = -J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j$, the system is invariant under a global rotation of all spins. As the volume influenced by the symmetry operation occupies a D - dimensional region, we have that $d = D$.

4. Examples of intermediate symmetries

a) Orbitals- In transition metal (TM) systems on cubic lattices, each TM atom is surrounded by an octahedral cage of oxygens. Crystal fields lift the degeneracy of the five 3d orbitals of the TM to two higher energy e_g levels ($|d_{3z^2-r^2}\rangle$ and $|d_{x^2-y^2}\rangle$)

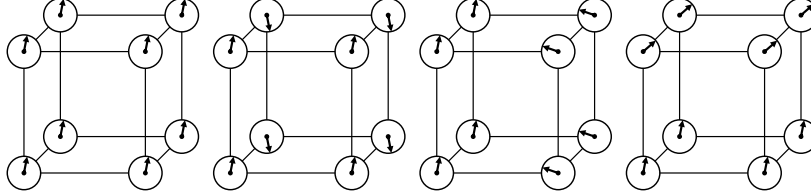


Fig. 1. From Refs^{3 4}. The symmetries of Eq.(9) applied on the uniform state (at left).

and to three lower energy t_{2g} levels ($|d_{xy}\rangle$, $|d_{xz}\rangle$, and $|d_{yz}\rangle$). A super-exchange calculation leads to the Kugel-Khomskii Hamiltonian^{12 13}

$$H = \sum_{\langle r, r' \rangle} H_{orb}^{r, r'} (\vec{S}_r \cdot \vec{S}_{r'} + \frac{1}{4}). \quad (5)$$

Here, \vec{S}_r denotes the spin of the electron at site r and $H_{orb}^{r, r'}$ are operators acting on the orbital degrees of freedom. For TM-atoms arranged in a cubic lattice,

$$H_{orb}^{r, r'} = J(4\hat{\pi}_r^\alpha \hat{\pi}_{r'}^\alpha - 2\hat{\pi}_r^\alpha - 2\hat{\pi}_{r'}^\alpha + 1), \quad (6)$$

where $\hat{\pi}_r^\alpha$ are orbital pseudospins and $\alpha = x, y, z$ is the direction of the bond $\langle r, r' \rangle$.

(i) In the e_g compounds,

$$\hat{\pi}_r^x = \frac{1}{4}(-\sigma_r^z + \sqrt{3}\sigma_r^x), \quad \hat{\pi}_r^y = \frac{1}{4}(-\sigma_r^z - \sqrt{3}\sigma_r^x), \quad \hat{\pi}_r^z = \frac{1}{2}\sigma_r^z. \quad (7)$$

This also defines the orbital only “120°-Hamiltonian” given by

$$H_{orb} = J \sum_{r, r'} \sum_{\alpha=x, y, z} \hat{\pi}_r^\alpha \hat{\pi}_{r+\hat{e}_\alpha}^\alpha. \quad (8)$$

Jahn-Teller effects in e_g compounds also lead, on their own, to orbital interactions of the 120°-type.¹³ The “120° model” model of Eqs.(7,8) displays discrete ($d = 2$) $[Z_2]^{3L}$ gauge-like symmetries (corresponding to planar Rubick’s cube like reflections about internal spin directions- Fig.(1)). The symmetry operators O^α are^{2 3 4}

$$O^\alpha = \prod_{r \in P_\alpha} \hat{\pi}_r^\alpha. \quad (9)$$

Here, $\alpha = x, y, z$ and P_α may denote any plane orthogonal to the cubic \hat{e}_α axis.

(ii) In the t_{2g} compounds (e.g., LaTiO_3), we have in H_{orb} of Eq.(8)

$$\hat{\pi}_r^\alpha = \frac{1}{2}\sigma_r^\alpha. \quad (10)$$

This is called the *orbital compass* model. The symmetries of this Hamiltonian are given by Eqs.(9, 10). Rotations of individual lower-dimensional planes about an axis orthogonal to them leave the system invariant. The two dimensional orbital compass model is given by Eqs.(8,10) on the square lattice with $\alpha \in \{x, z\}$ and displays $d = 1$ Z_2 symmetries (wherein the planes P of Eq.(9) become lines).

b) Spins in transition metal compounds- Following ¹⁴, we label the three t_{2g} states $|d_{yz}\rangle, |d_{xz}\rangle, |d_{xy}\rangle$ by $|X\rangle, |Y\rangle$, and $|Z\rangle$. In the t_{2g} compounds, hopping is prohibited via intermediate oxygen p orbitals between any two electronic states of orbital flavor α ($\alpha = X, Y$, or Z) along the α axis of the cubic lattice (see Fig.2). As a consequence, as noted in ¹⁴, a uniform rotation of all spins, whose electronic orbital state is $|\alpha\rangle$, in any given plane (P) orthogonal to the α axis $c_{i\alpha\sigma}^\dagger = \sum_\eta U_{\sigma,\eta}^{(P)} d_{i\alpha\eta}^\dagger$ with σ, η the spin directions, leaves Eq.(5) invariant. The net spin of the electrons of orbital flavor $|\alpha\rangle$ in any plane orthogonal to the cubic α axis is conserved. Here, we have $d = 2$ $SU(2)$ symmetries

$$\hat{O}_{P;\alpha} \equiv [\exp(i\vec{S}_P^\alpha \cdot \vec{\theta}_P^\alpha)/\hbar], \quad [H, \hat{O}_{P;\alpha}] = 0, \quad (11)$$

with $\vec{S}_P^\alpha = \sum_{i \in P} \vec{S}_i^\alpha$, the sum of all the spins \vec{S}_i^α in the orbital state α in any plane P orthogonal to the direction α (see Fig.2).

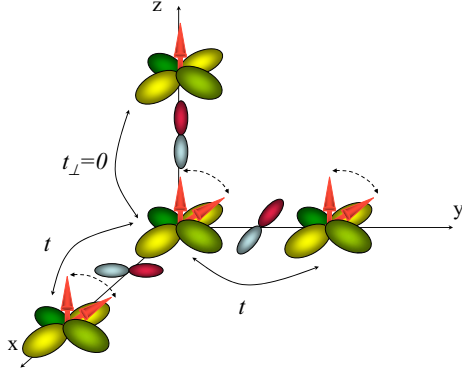


Fig. 2. From Ref.². The anisotropic hopping amplitudes leading to the Kugel-Khomskii (KK) Hamiltonian. Similar to Ref.¹³ the four lobed states denote the $3d$ orbitals of a transition metal while the intermediate small p orbitals are oxygen orbital through which the super-exchange process occurs. The dark and bright shades denote positive and negative regions of the orbital wavefunction. Due to orthogonality with intermediate oxygen p states, in any orbital state $|\alpha\rangle$ (e.g. $|Z\rangle \equiv |d_{xy}\rangle$ above), hopping is forbidden between sites separated along the cubic α (Z above) axis. The ensuing super-exchange (KK) Hamiltonian exhibits a $d = 2$ $SU(2)$ symmetry corresponding to a uniform rotation of all spins whose orbital state is $|\alpha\rangle$ in any plane orthogonal to the cubic direction α .

c) Superconducting arrays: A Hamiltonian for superconducting $(p + ip)$ grains (e.g. of Sr_2RuO_4) on a square grid, was recently proposed, ¹⁵

$$H = -K \sum_{\square} \sigma^z \sigma^z \sigma^z \sigma^z - h \sum_{\mathbf{r}} \sigma_{\mathbf{r}}^x. \quad (12)$$

Here, the four spin product is the product of all spins common to a given plaquette \square . The spins reside on the vertices on the plaquette (not on its bonds as gauge fields).

These systems have ($d = 1$ Z_2) symmetries similar to those of the two-dimensional orbital compass model. With P any row or column, $\hat{O}_P = \prod_{\vec{r} \in P} \sigma_{\vec{r}}^x$, $[H, \hat{O}_P] = 0$.

d) Other systems: In ¹⁶ ¹⁷ similar symmetries were found in frustrated spin systems. Ring exchange Bose metals, in the absence of nearest neighbor boson hopping, exhibit $d = 1$ symmetries. ¹⁸ Continuous *sliding symmetries* of Hamiltonians (actions) invariant under arbitrary deformations along a transverse direction,

$$\phi(x, y) \rightarrow \phi(x, y) + f(y), \quad (13)$$

appear in many systems. Amongst others, such systems were discovered in works on Quantum Hall liquid crystalline phases, ¹⁹ ²⁰ a number of models of lipid bilayers with intercalated DNA strands, ²¹ and sliding Luttinger liquids. ²²

5. A theorem on dimensional reduction

The absolute mean value of any local quantity (involving only a finite number of fields) which is not invariant under a d -dimensional symmetry group G of the D -dimensional Hamiltonian H is equal or smaller than the absolute mean value of the same quantity computed for a d -dimensional Hamiltonian \bar{H} which is globally invariant under G and preserves the range of the interactions. ² Non invariant means that the quantity under consideration, $f(\phi_i)$, has no invariant component:

$$\sum_k f[\mathbf{g}_{ik}(\phi_i)] = 0. \quad (14)$$

For a continuous group, this is replaced by $\int f[\mathbf{g}_i(\phi_i)] d\mathbf{g} = 0$. To determine if spontaneous symmetry breaking occurs, we compute

$$\langle f(\phi_i) \rangle = \lim_{h \rightarrow 0} \lim_{N \rightarrow \infty} \langle f(\phi_i) \rangle_{h, N}, \quad (15)$$

where $\langle f(\phi_i) \rangle_{h, N}$ is the mean value of $f(\phi_i)$ computed on finite lattice of N sites and in the presence of a symmetry breaking field h . Since $\Lambda = \bigcup_l \mathcal{C}_l$, the site \mathbf{i} belongs at least to one set \mathcal{C}_j . It is convenient to rename the fields in the following way: $\phi_i = \psi_i$ if $\mathbf{i} \notin \mathcal{C}_j$ and $\phi_i = \eta_i$ if $\mathbf{i} \in \mathcal{C}_j$. The mean value $\langle f(\phi_i) \rangle_{h, N}$ is given by:

$$\begin{aligned} \langle f(\phi_i) \rangle_{h, N} &= \frac{\sum_{\{\phi_i\}} f(\phi_i) e^{-\beta H(\phi)} e^{-\beta h \sum_i \phi_i}}{\sum_{\{\phi_i\}} e^{-\beta H(\phi) - \beta h \sum_i \phi_i}} = \\ &= \frac{\sum_{\{\psi_i\}} z_{\{\psi\}} e^{-\beta h \sum_{i \notin \mathcal{C}_j} \psi_i} \left[\frac{\sum_{\{\eta_i\}} f(\eta_i) e^{-\beta H(\phi) - \beta h \sum_{i \in \mathcal{C}_j} \eta_i}}{z_{\{\psi\}}} \right]}{\sum_{\{\psi_i\}} z_{\{\psi\}} e^{-\beta h \sum_{i \notin \mathcal{C}_j} \psi_i}} \\ &\text{with, } z_{\{\psi\}} = \sum_{\{\eta_i\}} e^{-\beta H(\psi, \eta) - \beta h \sum_{i \in \mathcal{C}_j} \eta_i}. \end{aligned} \quad (16)$$

From Eq.(16):

$$|\langle f(\phi_i) \rangle_{h, N}| \leq \left| \frac{\sum_{\{\eta_i\}} f(\eta_i) e^{-\beta H(\bar{\psi}, \eta) - \beta h \sum_{i \in \mathcal{C}_j} \eta_i}}{z_{\{\bar{\psi}\}}} \right|, \quad (17)$$

where $\{\bar{\psi}\}$ is the particular configuration of fields $\psi_{\mathbf{i}}$ that maximizes the expression between brackets in Eq.(16). $H(\bar{\psi}, \eta)$ is a d -dimensional Hamiltonian for the field variables η which is invariant under the *global* symmetry group G_j of transformations \mathbf{U}_{jk} over the field η . We can define $\bar{H}(\eta) \equiv H(\bar{\psi}, \eta)$. The range of the interactions between the η -fields in $\bar{H}(\eta)$ is clearly the same as the range of the interactions between the ϕ -fields in $H(\phi)$. This completes the demonstration of our theorem. Note that the “frozen” variables $\bar{\psi}_{\mathbf{i}}$ act like external fields in $\bar{H}(\eta)$ which do not break the global symmetry group of transformations \mathbf{U}_{jk} .

*Corollary I: Elitzur’s theorem.*¹⁰ Any local quantity (i.e. involving only a finite number of fields) which is not invariant under a local (or $d = 0$) symmetry group has a vanishing mean value at any finite temperature. This is a direct consequence of Eq.(16) and the fact that $\bar{H}(\eta)$ is a zero-dimensional Hamiltonian.²

*Corollary II.*² A local quantity which is not gauge invariant under a one-dimensional intermediate symmetry group has a vanishing mean value at any finite temperature for systems with finite range interactions. This is a consequence of Eq.(17) and the absence of spontaneous symmetry breaking in one-dimensional Hamiltonians such as $\bar{H}(\eta) \equiv H(\bar{\psi}, \eta)$ with interactions of finite range and strength. Here, $f(\eta_{\mathbf{i}})$ is a non-invariant under the global symmetry group G_j [see Eq.(14)].

*Corollary III.*² In finite range systems, local quantities not invariant under continuous two-dimensional symmetries have a vanishing mean value at any finite temperature. This results from [Eq.(17)] with the Mermin-Wagner theorem²³:

$$\lim_{h \rightarrow 0} \lim_{N \rightarrow \infty} \frac{\sum_{\{\eta_{\mathbf{i}}\}} f(\eta_{\mathbf{i}}) e^{-\beta H(\bar{\psi}, \eta) - \beta h \sum_{\mathbf{i} \in C_j} \eta_{\mathbf{i}}}}{Z_{\{\bar{\psi}\}}} = 0. \quad (18)$$

We invoked that G_j is a continuous symmetry group of $\bar{H}(\eta) = H(\bar{\psi}, \eta)$, $f(\eta_{\mathbf{i}})$ is a non-invariant quantity for G_j [see Eq.(14)], and $\bar{H}(\eta)$ is a two-dimensional Hamiltonian that only contains finite range interactions.

The generalization of this theorem to the quantum case is straightforward if we choose a basis of eigenvectors of the local operators linearly coupled to the symmetry breaking field h . Here, the states can be written as a direct product $|\phi\rangle = |\psi\rangle \otimes |\eta\rangle$. Eq.(17) is re-obtained with the sums replaced by traces over the states $|\eta\rangle$:

$$|\langle f(\phi_{\mathbf{i}}) \rangle_{h,N}| \leq \frac{\text{Tr}_{\{\eta\}} f(\eta_{\mathbf{i}}) e^{-\beta H(\bar{\psi}, \eta) - \beta h \sum_{\mathbf{i} \in C_j} \eta_{\mathbf{i}}}}{\text{Tr}_{\{\eta\}} e^{-\beta H(\bar{\psi}, \eta) - \beta h \sum_{\mathbf{i} \in C_j} \eta_{\mathbf{i}}}}, \quad (19)$$

In this case, $|\bar{\psi}\rangle$ corresponds to one particular state of the basis $|\psi\rangle$ that maximizes the right side of Eq.(19). Generalizing standard proofs, e.g.²⁴, we find a zero temperature quantum extension of Corollary III in the presence of a gap:

*Corollary IV.*² If a gap exists in a system possessing a $d \leq 2$ dimensional continuous symmetry in its low energy sector then the expectation value of any local quantity not invariant under this symmetry, strictly vanishes at zero temperature. Though local order cannot appear, multi-particle (incl. topological) order can exist.

Corollary V. The absolute values of non-symmetry invariant correlators $|G| \equiv |\langle \prod_{i \in \Omega_j} \phi_i \rangle|$ with $\Omega_j \subset C_j$ are bounded from above by absolute values of the same correlators $|G|$ in a d dimensional system defined by C_j in the presence of transverse non-symmetry breaking fields. If no resonant terms appear in the lower dimensional spectral functions (due to fractionalization), this allows for fractionalization of non-symmetry invariant quantities in the higher dimensional system.

6. Consequences of the theorem

(a) *Spin nematic order in t_{2g} systems:* If the KK Hamiltonian (Eq.(5)) captures the spin physics of t_{2g} compounds, then no magnetization can exist at finite temperature² due to the continuous $d = 2$ symmetries¹⁴ that it displays (Eq.(11)).²
¹⁴ Empirically, low temperature magnetization is detected. Thus, the KK Hamiltonian of Eq.(5) may be augmented by other interactions which lift this symmetry. The simplest quantities invariant under these symmetries are nematic order parameters. In the presence of orbital ordering in the $|\alpha\rangle$ state, superpositions of $\langle \vec{S}_r \cdot \vec{S}_{r+n\hat{e}_\eta} \rangle$, with $\eta = x, y, z$ where $\eta \neq \alpha$ and n an integer, need not vanish. If the KK Hamiltonian embodies the dominant contribution to the spin physics, nematic order might persist to far higher temperatures than the currently measured magnetization.²

(b) *Orbital order:* The orbital only Hamiltonians discussed earlier exhibit a $d = 2$ discrete Z_2 symmetry. The theorem² allows such symmetries to be broken. Indeed, as we will review shortly, in these orbital only Hamiltonians, order already appears at the classical level- a tendency which may be enhanced by quantum fluctuations.

(c) *Nematic orbital order in two dimensional $(p+ip)$ superconducting arrays and two dimensional orbital systems:* The two dimensional $(p+ip)$ superconducting arrays of Eq.(12) exhibit a $d = 1$ Z_2 symmetry. As these symmetries cannot be broken, no magnetization can arise, $\langle \sigma_\alpha \rangle = 0$. The simplest symmetry allowed order parameter is of the nematic type which is indeed realized classically.^{3 4 25}

(d) *Fractionalization in spin and orbital systems:* Corollary (V) allows for fractionalization in quantum systems where $d = 1, 2$. It enables symmetry invariant quasi-particles excitations to *coexist* with non-symmetry invariant fractionalized excitations. Fractionalized excitations may propagate in $d_s = D - d$ dimensional regions (with D the spatial dimensionality of the system). Examples afforded by several frustrated spin models where spinons may drift along lines ($d_s = 1$) on the square lattice¹⁶ and in $d_s = D$ dimensional regions on the pyrochlore lattice.¹⁷

(e) *Absence of charge order:* In systems, such as quantum Hall smectics, in which the system is invariant to the charge density variations of Eq.(13), we have $\langle \phi \rangle = 0$.

7. Order by disorder in symmetry allowed instances

When symmetry breaking is allowed (e.g. the two dimensional Ising symmetry (Eq.(9)) of the 120° Hamiltonian), order often transpires by a fluctuation driven mechanism (“order by disorder”).²⁶ Although several states may appear to be equally valid candidate ground state, fluctuations can stabilize those states which

have the largest phase space volume for low energy fluctuations about them. These differences are captured in values of the free energies for fluctuations about the contending states. Classically, fluctuations are driven by thermal effects. Quantum tunneling processes may fortify such tendencies.

If the Pauli matrices σ in Eq.(8) are replaced by the spin S generators and the limit $S \rightarrow \infty$ is taken then we will obtain the classical 120° model. Here, the *free energy* has strict minima for six uniform orientations^{3 4} $\vec{S}_i = \pm S\hat{a}, \vec{S}_i = \pm S\hat{b}, \vec{S}_i = \pm S\hat{c}$. Out of the exponentially large number of ground states (supplanted by an additional global $U(1)$ rotational symmetry which emerges in the ground state sector), only six are chosen. Interfaces between uniform states (such as that borne by the application of d=2 Z_2 reflections on a uniform state, see fig.(1)) leads to a surface tension additive in the number of symmetry operations. Being of an entropic origin, the surface tension between various uniform domains is temperature independent and does not diverge at low temperatures.^{3 4} Orbital order already appears within the classical (formally, $S \rightarrow \infty$) limit^{3 4} and is not exclusively reliant on subtle quantum zero point fluctuations (captured by $1/S$ calculations) for its stabilization. Indeed, orbital order is detected up to relatively high temperatures ($\mathcal{O}(100K)$).²⁷

8. Dualities as space-time reflections

Explicit operator representations show that the two dimensional variant of the orbital compass model (Eqs.(8, 10) is dual to the Xu-Moore model of $(p + ip)$ superconducting arrays (Eq.(12)).⁵ We now examine this duality in the discrete Euclidean path integral formulation. This examination illustrates how geometrical reflections may lead to dualities.⁵ In a basis quantized along σ^z , the zero temperature Euclidean action of the two dimensional orbital compass model is

$$S = -K_x \sum_{\square \in (x\tau) \text{ plane}} \sigma_{r,\tau}^z \sigma_{r,\tau+\Delta\tau}^z \sigma_{r+\hat{e}_x,\tau}^z \sigma_{r+\hat{e}_x,\tau+\Delta\tau}^z - (\Delta\tau) J_z \sum_r \sigma_r^z \sigma_{r+e_z}^z. \quad (20)$$

A schematic of this action in Euclidean space-time is shown in Fig.(3). If we relabel the axes and replace the spatial index x with the temporal index τ , we will immediately find the classical action corresponding to the the Hamiltonian of Eq.(12) depicting $p + ip$ superconducting grains in a square grid. This suggests that the planar orbital compass system and the $(p + ip)$ Hamiltonian (Eq.(12)) are dual to each other as indeed occurs at all temperatures. Strong-weak coupling dualities that these Hamiltonians (and others) display can be similarly established.⁵

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References

1. X-G. Wen, *Quantum Field Theory of Many-Body Systems* (Oxford University Press, Oxford, 2004), and references therein.
2. C. D. Batista and Z. Nussinov, Phys. Rev. B **72**, 045137 (2005)

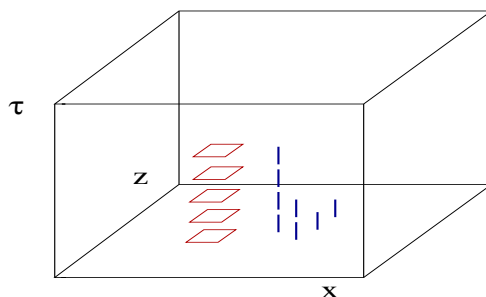


Fig. 3. From Ref.⁵. The classical Euclidean action corresponding to the Hamiltonian of Eq.(12) at zero temperature in a basis quantized along the σ^z direction. The transverse field leads to bonds parallel to the imaginary time axis while the plaquette interactions become replicated along the imaginary time axis. Taking an equal time slice of this system, we find the four spin term of Eq.(12) and the on-site magnetic field term. If we interchange τ with z , we find the planar orbital compass model in the basis quantized along the σ^x direction.

3. Z. Nussinov, M. Biskup, L. Chayes, J. v. d. Brink, Europhysics Letters, **67**, 990 (2004)
4. M. Biskup, L. Chayes, Z. Nussinov, Comm. Math. Phys., **255**, 253 (2005)
5. Z. Nussinov and E. Fradkin, Phys. Rev. B **71**, 195120 (2005)
6. J. B. Kogut, Reviews of Modern Physics, **51**, 659 (1979)
7. E. Fradkin and S. H. Shenker, Phys. Rev. D **19**, 3682 (1979)
8. F. Wegner, J. Math. Phys. **12**, 2259 (1971); K. G. Wilson, Phys. Rev. D **7**, 2911 (1974); J. Kogut and L. Susskind, Phys. Rev. D **11**, 395 (1975); R. Balian, J. M. Drouffe, and C. Itzykson, Phys. Rev. D **11**, 2098 (1975)
9. Z. Nussinov, Phys. Rev. D **72**, 054509 (2005)
10. S. Elitzur, Phys. Rev. D **12**, 3978 (1975)
11. Y. Aharonov and D. Bohm, Phys. Rev. **11**, 485 (1959)
12. K. I. Kugel and D. I. Khomskii, Sov. Phys. JETP **37**, 725 (1973)
13. J. v. d. Brink, G. Khaliullin and D. Khomskii, *Orbital effects in manganites*, T. Chatterij (ed.), *Colossal Magnetoresistive Manganites*, Kluwer Academic Publishers (2002)
14. A. B. Harris et al., Phys. Rev. Lett. **91**, 087206 (2003)
15. C. Xu and J. E. Moore, Phys. Rev. Lett. **93**, 047003 (2004).
16. C. D. Batista and S. A. Trugman, Phys. Rev. Lett. **93**, 217202 (2004).
17. Z. Nussinov, C. D. Batista, B. Normand, and S. A. Trugman, cond-mat/0602528
18. A. Paramekanti, L. Balents, and M. P. A. Fisher, Phys. Rev. B **66**, 054526 (2002)
19. M. J. Lawler and E. Fradkin, Phys. Rev. B. **70**, 165310 (2004)
20. L. Radzihovsky and A. T. Dorsey, Phys. Rev. Lett., **88**, 216802 (2002)
21. C. S. O'Hern and T. C. Lubensky, Phys. Rev. E **58**, 5948 (1998)
22. V. J. Emery et al., Phys. Rev. Lett., **85**, 2160 (2000)
23. N. D. Mermin and H. Wagner, Phys. Rev. Lett. **17**, 1133 (1966).
24. Assa Auerbach, "Interacting Electrons and Quantum Magnetism", Springer-Verlang 1994 (chapter 6, in particular)
25. A. Mishra et al., Phys. Rev. Lett. **93**, 207201 (2004)
26. E.F. Shender, Sov. Phys. JETP **56**, 178 (1982); C.L. Henley, Phys. Rev. Lett. **62**, 2056 (1989).
27. Y. Murakami *et al*, Phys. Rev. Lett. **80**, 1932 (1998); Y. Endoh *et al*, Phys. Rev. Lett. **82**, 4328 (1999); S. Ishihara and S. Maekawa, Phys. Rev. Lett. **80**, 3799 (1998); I.S. Elfimov, V.I. Anisimov and G.A. Sawatzky, Phys. Rev. Lett. **82**, 4264 (1999). Y. Tokura and N. Nagaosa, Science **288**, 462 (2000), and references therein.